The generalized uncertainty principle in the presence of extra dimensions

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We argue that in the Generalized Uncertainty Principle (GUP) model, the parameter β_0 whose square root, multiplied by Planck length ℓ_p, approximates the minimum measurable distance, varies with energy scales. Since minimal measurable length and extra dimensions are both suggested by quantum gravity theories, we investigate models based on GUP and one extra dimension, compactified with radius ρ. We obtain an inspiring relation $\sqrt{\beta_0}\ell_p/\rho \sim \mathcal{O}(1)$. This relation is also consistent with predictions at Planck scale and usual quantum mechanics scale. We also make estimations on the application range of the GUP model. It turns out that the minimum measurable length is exactly the compactification radius of the extra dimension.

PACS numbers:

arXiv:0909.3635v2 [hep-th] 24 Sep 2009

One of the predictions shared by various quantum theories of gravity is the existence of a minimum measurable distance, proportional to the Planck length $\ell_p \sim (10^{-33}\text{cm})$ [1], at the Planck scale. This distance is caused by the nonlinear interactions between gravity and matter. Uncertainty in the momentum of a particle induces uncertainty on the geometry which leads to an extra uncertainty to the position of the particle. Equivalently speaking, there exists a cutoff on the energy scale, serving as an ultimate ultraviolet regulator. One can refer to [2] for a review of the origins of the minimum measurable distance from various scenarios. In the spirit of the Hierarchy problem, one would naturally expect that remnants of the minimum measurable distance phenomena show up in effective theories at an up-to-determined intermediate scale. In this letter, when modeled with an extra dimension, we show that the minimum measurable distance is the same scale as the radius of the compactified extra dimension.

Some realizations of the minimum measurable distance are proposed. One of the most important models is the generalized uncertainty principle (GUP), derived from the modified fundamental commutator[3, 4, 5, 6]:

$$[x, p] = if(p), \tag{1}$$

where $f(p)$ is a positive function with $f(0) = 1$ [19] to reproduce the usual fundamental commutator at the low energy limit. We set $\hbar = c = 1$ in this letter. The Taylor expansion of $f(p)$ around $p = 0$ is:

$$f(p) = 1 + \beta p^2 + \mathcal{O}(p^4), \tag{2}$$

with $\beta = \beta_0 \ell_p^2 = \beta_0/M_p^2$ and β_0 is a dimensionless number. $M_p \sim 10^{19}$GeV is the Planck mass. When restricted to low energy effective theories, only the linear term of β is kept:

$$[x, p] = i(1 + \beta p^2). \tag{3}$$

With this generalization, one can easily derive the GUP:

$$\Delta x \Delta p \geq \frac{1}{2}\left[1 + \beta(\Delta p)^2\right], \tag{4}$$

which in turn gives the minimum measurable distance $\Delta x \geq \Delta x_{\min} = \sqrt{\beta} = \sqrt{\beta_0}\ell_p$. The tensorial generalization of (3) to higher dimensions is represented by [20]

$$[x_i, p_j] = i(\delta_{ij} + \beta\delta_{ij}p^2 + 2\beta p_i p_j), \tag{5}$$

accompanied with $[x_i, x_j] = [p_i, p_j] = 0$. Then the multidimensional version of GUP is

$$\Delta x_i \Delta p_i \geq \frac{1}{2}\Big[1 + \beta\big((\Delta p)^2 + \langle p\rangle^2\big) + 2\beta\big(\Delta p_i^2 + \langle p_i\rangle^2\big)\Big], \tag{6}$$

where $i = 1, 2, 3$ and $p^2 = \sum_{i=1}^3 p_i p_i$. The minimum observable length for every direction is $\Delta(x_i)_{\min} \sim \sqrt{\beta_0}\ell_p$. It should be emphasized that GUP is an low energy approximation, capturing merely one of the features of the physics at Planck scale. There exists no substantial reason to expect that parameters in GUP are equal to those in full quantum theories of gravity.

The dimensionless number $\beta_0 = \beta/\ell_p^2 = \beta M_p^2$ plays an important role. From eqn. (4), in the GUP model, it defines an intermediate energy scale where the gravity goes on stage,

$$M_I \sim M_p/\sqrt{\beta}_0. \tag{7}$$

Normally, β_0 is assumed to be of order unity. This makes sense near the Planck scale, as indicated by various quantum theories of gravity. However, when constructing low energy effective theories, GUP for instance, this assumption has no firm basis. If $\beta_0 \sim 1$, one obtains $M_I \sim M_p$. Nevertheless, once approaching the Planck scale, the effective theory loses its effectiveness and should be replaced by full quantum theories of gravity. On the other

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hand, people tend to believe that there exist some new physics at an intermediate scale M_I ($< M_p$), retaining some features of the physics at the Planck scale. It is tempting to conjecture that the minimum measurable length emerges already at some M_I. At this intermediate scale, β_0 is not order of unity anymore but a large number determined by eqn. (7). The change of β_0 from order of unity to a large number may be caused by some other suppressed features of the full quantum theories of gravity.

The variation of β_0 could be justified by another consideration. In physics, the existence of invariant dimensionless parameters spoils the uniqueness of a theory. An immediate question is that why our universe picks up a set of particular numbers. The presence of dimensionless parameters either indicates that the theory is an effective one or the parameters are not really invariant with energy scales. Since β_0 originates from a full quantum gravity theory, which is believed to be a final, unique theory, it is hard to think that β_0 keeps fixed in all scales.

Therefore, we conjecture that β_0 runs with the energy scale. The story is not odd to us in quantum field theories (QFT). From the renormalization group point of view, a cutoff M_I of an effective theory defines a grainy space with spacing $\Delta x = f(\frac{1}{M_I})$, where f is certain polynomial function. This Δx is believed to be caused by quantum fluctuation of gravity and then could be an realization of the fundamental minimum measurable length in a QFT. For this reason, the upper limit of β_0 may exceed that set by the weak scale: $M_{\mathrm{EW}} \sim 10^2$GeV. However, β_0 cannot increase forever to conflict with experimental results in the usual quantum mechanics regime.

Consequently, from the weak scale, one obtains the roughest estimation of the upper bound $\beta_0 < 10^{34}$. A better result is presented by the conjectured genuine fundamental scale of ADD model [9] $M_I = M_f = 10^4$Gev, where the effects of gravity are assumed to be easily visible, conjectured in that model. With this scale,

$$\beta_0 \lesssim (M_p/M_f)^2 = 10^{30}. \tag{8}$$

This upper bound is expected to be verified by LHC in the near future. An up to date estimation $\beta_0 < 10^{21}$ is given in [6] with some lax assumptions. In their calculation of some examples, the authors also showed that GUP effect is unobservable with $\beta_0 \sim 1$, consistent with our arguments given above. This estimation indicates that the gravity becomes important at $M_I \sim 10^9$GeV, much larger than the ADD scale. Though out of the scope of terrestrial experiments, it is indeed a reasonable intermediate scale. In this letter, we take an upper bound based on the precision measurement of Lamb shift, given in [6],

$$\beta_0 < 10^{36}. \tag{9}$$

The corresponding scale is $M_I \sim 10$GeV. Joyfully it is in the scope of current experiments. More accurate measurement is anticipated in the near future to visualize the predictions of GUP or to bring down the upper bound.

On the other hand, though $\beta_0 \sim 1$ near the Planck scale, in a specific model, a lower bound on β_0 greater than unity could exist due to the application range of the model. In the case of GUP, a nice lower limit arises from the Grand Unification Theory (GUT) scale in the Minimal Supersymmetric Standard Model (MSSM), where the gravity becomes strong

$$M_I < M_{\mathrm{GUT}} \sim 10^{-3} M_p \Rightarrow \beta_0 > 10^6. \tag{10}$$

At present, the existence of extra dimensions is widely accepted by theorists. The story can be traced back to 1920's by the work of Kaluza-Klein (KK). String theory demands ten dimensional spacetime to have the anomalies canceled. Since the mid-1990's, this subject has attracted large number of works. Several paradigms [8, 9, 10, 11, 12, 13, 14] are proposed and significant progress is achieved. A good review is [15] and references therein.

Both minimum measurable distance and extra dimension are suggested by quantum theories of gravity. It is therefore instructive to combine them together in one effective theory. This reflection may shed light on some information about β_0 or the scales of the extra dimensions.

Guided with this idea, in this letter, based on GUP, we investigate quantum systems with one extra dimension w compactified on S^1 of radius ρ. We choose the KK compactification in this letter. Alternative constructions based on minimum measurable length and large extra dimensions are discussed in [16] with various applications. Discussion on holographic counting problems within the framework of extra dimensions and GUP can be found in [17] with references. To implement the generalized commutators (5), one defines

$$\begin{aligned} x_i &= x_{0i}, \\ p_i &= p_{0i}\left(1 + \beta p_0^2\right), \quad p_{0i} = -i\frac{d}{dx_{0i}}, \end{aligned} \tag{11}$$

where $p_0^2 = \sum p_{0j} p_{0j}$ and $[x_{0i}, p_{0j}] = i\delta_{ij}$, the usual canonical operators. One can easily show that to the first order of β, (5) is guaranteed. For a quantum system described by

$$H = \frac{p^2}{2m} + V(\vec{x}), \tag{12}$$

the modifications in (11) can be treated as a perturbation:

$$H = H_0 + H_1 = \frac{p_0^2}{2m} + V(\vec{x}_0) + \frac{\beta}{m} p_0^4. \tag{13}$$

The first order correction is then given as

$$\begin{aligned} E_n^{(1)} &= \frac{\beta}{m}\langle n| p_0^4 |n\rangle = 4m\beta\left\langle \left(E_n^{(0)} - V\right)^2 \right\rangle, \\ &= 4m\beta\left[\left(E_n^{(0)}\right)^2 - 2E_n^{(0)}\langle V\rangle + \langle V^2\rangle\right]. \end{aligned} \tag{14}$$

We discuss the simplest scenario in this letter. We suppose that in the extra dimension, particles experience vanishing potential with periodic boundary conditions. In a followed paper, we consider a simple harmonic oscillator with non-zero potential in the extra dimensions [18]. Thus, in the unperturbed system H_0 in (13), the extra dimension contributes a term

$$\frac{\ell^2}{2m\rho^2}, \qquad \ell = 0, 1, 2\ldots \tag{15}$$

Therefore, the unperturbed eigenenergies in eqn. (14) should be replaced as

$$E_n^{(0)} \to E_{n\ell}^{(0)\prime} \equiv E_n^{(0)} + \frac{\ell^2}{2m\rho^2}, \tag{16}$$

to reflect the modification from the extra dimension. The total energy is then

$$E_{n\ell} = E_{n\ell}^{(0)\prime} + 4m\beta\left[\left(E_{n\ell}^{(0)\prime}\right)^2 - 2E_{n\ell}^{(0)\prime}\langle V\rangle + \langle V^2\rangle\right]. \tag{17}$$

Generically, $\langle V\rangle \simeq E_n^{(0)}$. Therefore, when analyzing the order behaviors, on the *rhs* of eqn. (17), the last two terms involving $\langle V\rangle$ and $\langle V^2\rangle$ in the bracket can be ignored. The spectra are approximated as

$$E_{n\ell} \simeq E_n^{(0)}\left(1+4m\beta E_n^{(0)}\right)+\left(1+8m\beta E_n^{(0)}+2\beta\frac{\ell^2}{\rho^2}\right)\frac{\ell^2}{2m\rho^2}. \tag{18}$$

Consistency with the observations in quantum mechanics imposes constraints on β_0 and ρ as follows:

$$4m\beta E_1^{(0)} \ll 1 \quad \text{and} \quad \frac{1}{2m\rho^2} \gg E_1^{(0)}, \tag{19}$$

or equivalently $\beta_0 \ll 10^{49}$ and $\rho \ll 10^{-9}$cm with the *ground state* scale $mE_1^{(0)} \simeq 10^{-11}\mathrm{GeV}^2$ of typical quantum mechanics systems. From the spectra eqn. (18), it is easy to see that the gravity will not stage until $E_{n0} \sim E_{11}$, requiring $E_n^{(0)} \sim \frac{1}{2m\rho^2}$. To make this condition possible, unbound up system is indicated in our model. We are going to see that this is a very large scale compared with the ground state one. It should be pointed out that the parameter m here is the *physical* mass determined by the scale, different from the one in ground state. Therefore, the scale where gravity becomes important is

$$E_{n0} \sim \frac{1}{2m\rho^2}\left(1 + 2\beta_0\frac{\ell_p^2}{\rho^2}\right). \tag{20}$$

The nice thing is that from the general analysis at eqn. (7), the scale triggering gravity is $M_I = M_p/\sqrt{\beta_0} = 1/\sqrt{\beta}_0\ell_p$. Thus, we have

$$\frac{1}{2M_I\rho^2}\left(1 + \frac{2}{M_I^2\rho^2}\right) \sim M_I, \tag{21}$$

where m has been replaced by the physical effective mass M_I. With some simple algebraic calculation, it is easy to show that $\frac{1}{2M_I^2\rho^2} + \left(\frac{1}{M_I^2\rho^2}\right)^2 \sim 1$. Therefore, one simple and inspiring relation arises:

$$M_I\rho \sim 1 \quad \text{or} \quad \beta = \beta_0\ell_p^2 \sim \rho^2. \tag{22}$$

Before discussing the physical significance of this relation, let us loose the effectiveness of our model to the extreme situations without much rigor. At the high energy end, Planck scale, T-duality sets the minimum of the compatification radius $\rho_{\min} \sim \sqrt{\alpha'} \sim \ell_p$, where α' is the regge slop. Then eqn. (22) gives $\beta_0 \sim 1$, agreeing with the prediction from various quantum gravity theories! In the low energy limit, no evidence of extra dimension is ever observed. Effectively, this means that ρ approaches zero. It is then happy to find that $\beta_0 \to 0$, consistent with the fact that space is continuously measurable. Though equation (22) is derived from an effective model, we showed that it also possesses features of the extreme conditions.

Bearing in mind that there is an upper bound $\beta_0 < 10^{36}$ given by experimental results, one gets $\rho < 10^{-15}$cm. Though this length is much smaller than the conjectured upper limit ($\sim 10^{-2}$cm) in ADD model, no conflict is found with any experiment verified theory. The lower limit of the radius is given by $\rho > 10^{-30}$cm via the GUT scale of MSSM as we argued. We group the results as follows:

$$10^6 < \beta_0 < 10^{36}, \quad 10^{-30}\mathrm{cm} < \rho < 10^{-15}\mathrm{cm}, \tag{23}$$

along with the scale $10\mathrm{GeV} < M_I < M_{\mathrm{GUT}}$. Hopfully, the ranges could be narrowed by better refined models.

It is amazing that from eqn. (22), one immediately finds that the minimum measurable distance is exactly the radius of the compact direction:

$$\Delta x_{\min} = \sqrt{\beta} = \sqrt{\beta}_0\ell_p \sim \rho. \tag{24}$$

Though it may look astonishing at first sight, one should not be really surprised by this coincidence. The minimum measurable length provides a cutoff to an effective theory while the compactification radius defines the scale of the theory. Our results imply that once an effective theory is constructed, its application range arises simultaneously and the upper bound is close to its defining scale. A more interesting implication of our derivations is that the existence of minimum measurable length is probably nothing but the exhibition of extra dimensions. It is still unclear if extra dimensions can be detected by usual instruments other than gravity-based devices. To our best knowledge, on the other hand, the minimum measurable length is unreachable by apparatus based on gauge particles.

To summarize, we argued that in GUP, an effective model, the dimensionless parameter β_0 runs with energy scales. After taking into account one compactified extra dimension, we showed that the parameter $\beta = \beta_0\ell_p^2$

is the same order of and proportional to the compactification radius ρ. The predictions are also consistent with results at Planck scale and usual quantum mechanics scale. Rough ranges of β_0, ρ and GUP scale were also presented. Finally, we showed that the minimum measurable length is precisely the compactification radius of the extra dimension. We employed the simplest model in this letter. We hope that better refined models based on our construction will reveal more low energy consequences of quantum gravity and offer more precise predictions, application range of GUP, for instance, in the near future. Extensions include more compactified extra dimensions corresponding different β's, other compactification paradigms like ADD or Sundrum-Randall models, introducing nonvanishing periodic potentials on the extra dimensions and so on. More thorough and detailed discussions on the running property of β_0 is anticipated. Determining the upper limit of β_0 is of particular interest. Probably, generalizing our construction to quantum field theory is of help.

Acknowledgments

We are grateful to Bo Feng, Peng Wang and Qing-Hai Wang for instructive discussions. We would also like to acknowledge useful conversations with X. Guo and D. Chen. This work is supported in part by NSFC (Grant No.10705008) and NCET.

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